Encoding Discrete Fourier Transform as Quantized Tensor Train

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Joint work with Miles Stoudenmire, Steve White, Michael Lindsey

Yuehaw Khoo group meeting, UChicago

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Background

Quantized tensor train (QTT) is a type of tensor network with linear geometry, which represents an underlying function in a multi-scale fashion.

It was found useful in many fields, including but not limited to:

- Representing bosonic systems [1]
- Image compression [2]
- Differential equations, e.g. Navier-Stokes [3], Vlasov-Poisson [4]
- Encoding orbitals for computational chemistry [5]

Also closely related to “entanglement structure” of quantum algorithms!

[1] Eric Jeckelmann, Steven R. White, *Density-matrix renormalization-group study of the polaron problem in the Holstein model*, Phys. Rev. B 57, 6376–6385 (1998)
[2] Jose I. Latorre, *Image compression and entanglement,* quant-ph/0510031 (2005)
[3] Nikita Gourianov, et.al., *A quantum-inspired approach to exploit turbulence structures*, Nature Computational Science 2, 30–37 (2022)
[4] Erika Ye, Nuno F. G. Loureiro, *Quantum-inspired method for solving the Vlasov-Poisson equations*, Phys. Rev. E 106, 035208 (2022)
[5] Nicolas Jolly, Yuriel Nunez Fernandez, Xavier Waintal, *Tensorized orbitals for computational chemistry*, 2308.03508 (2023)

Background

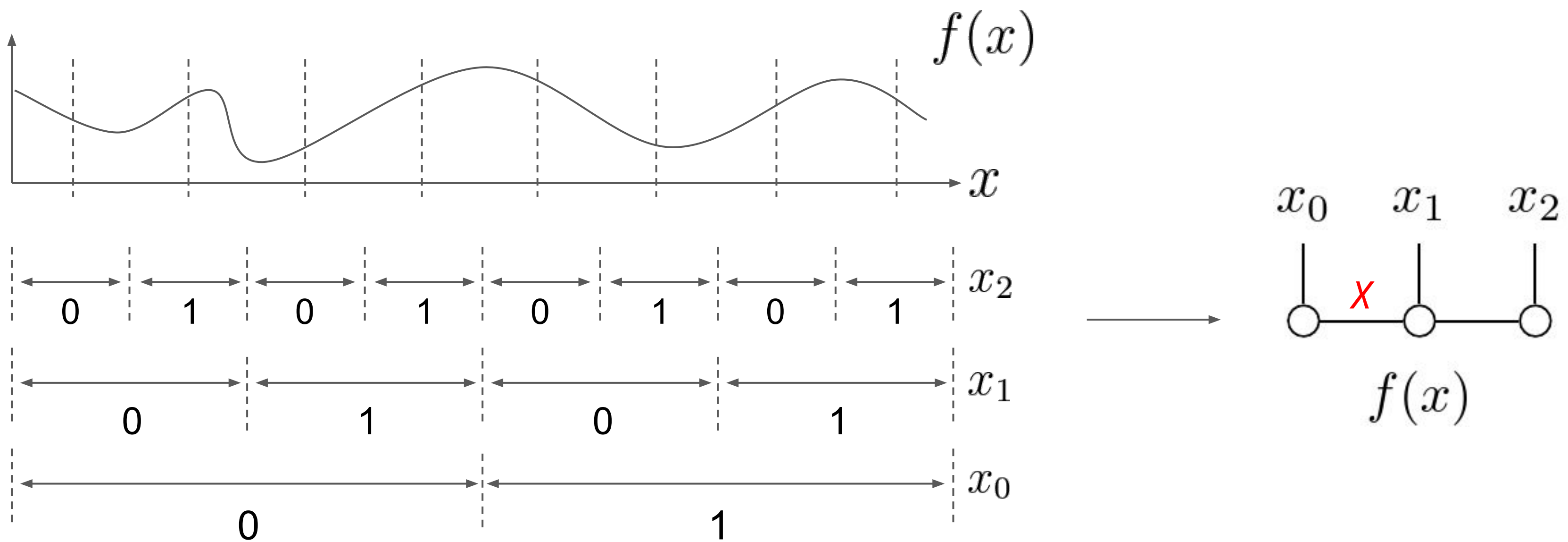
Recently it was found that discrete Fourier transform (DFT) can be encoded as an efficient QTT [1].

This could be useful for many QTT applications since DFT is a fundamental algorithm primitive.

This talk: recent results on constructing DFT in QTT format.

[1] JC, E.M. Stoudenmire, and Steven R. White, *Quantum Fourier Transform Has Small Entanglement*, PRX Quantum (2023) 4, 040318

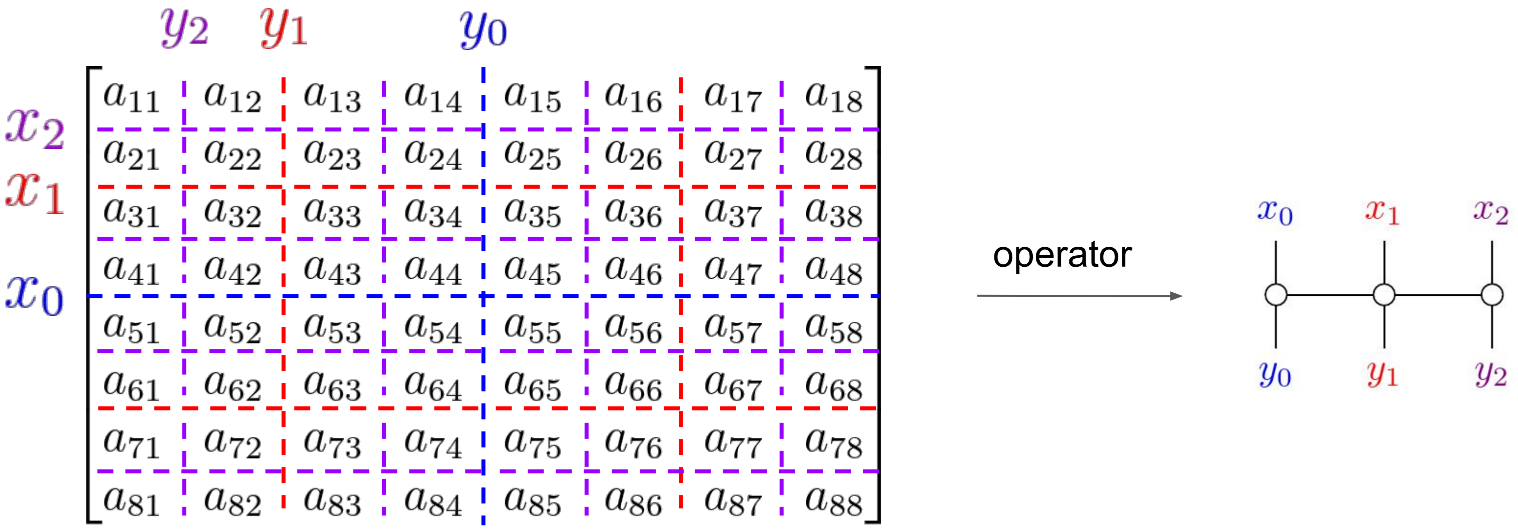
Quantized Tenor Train: 1D functions



Efficient criteria: bond-dimension χ = poly(n) or even χ = O(1)

Space savings: $2^n \rightarrow O(n\chi^2)$

Quantized Tenor Train: operators



Space savings: $2^{2n} \to O(n\chi^2)$

Similar scheme for multilinear map & high-dimensional functions

Discrete Fourier Transform

Defined as the following matrix, where $\omega = e^{-\frac{2\pi i}{N}}$ and $N = 2^n$

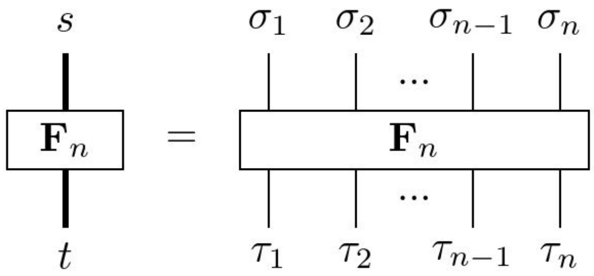
$$\mathbf{F}_n(s,t) = e^{-\frac{2\pi i s t}{N}} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

Discrete Fourier Transform

Can partition variables into binary strings:

$$\mathbf{F}_n(s,t) = e^{-\frac{2\pi i s t}{N}}$$

$$s = \sum_{k=1}^{n} 2^{n-k}\sigma_k, \quad t = \sum_{k=1}^{n} 2^{k-1}\tau_k$$

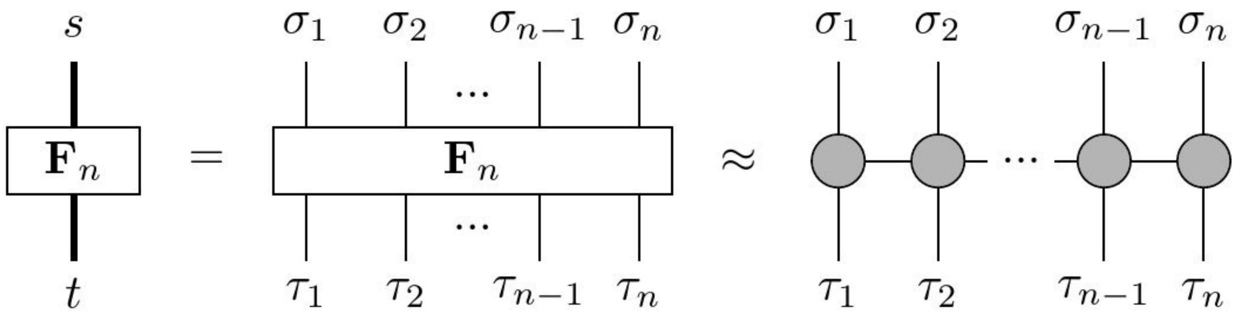


Notice: σ_1 is the most significant bit, but τ_1 is the least significant bit.

Why? Only this ordering provides us a low-rank QTT of DFT.
Normal ordering will give maximal QTT rank.

DFT in QTT

Main result: DFT can be approximated as low-rank QTT with small error



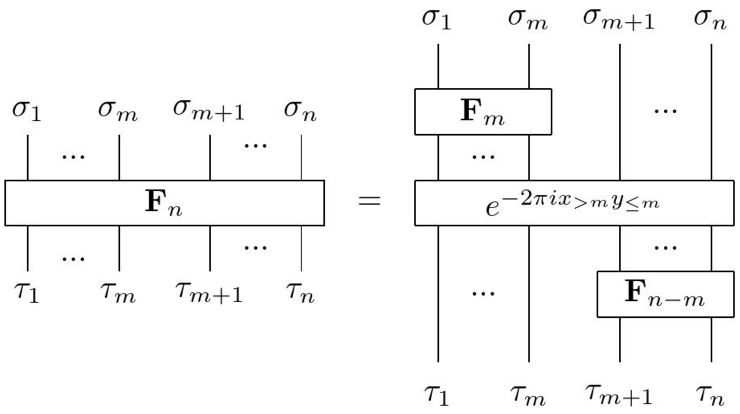
Proof by:

1. Schmidt decomposition (Frobenius norm error bound) [1]
2. Interpolative decomposition (Frobenius norm + entry-wise error bound + analytical construction) [2]

[1] JC, E.M. Stoudenmire, and Steven R. White, *Quantum Fourier transform has small entanglement*, PRX Quantum (2023) 4, 040318
[2] JC, Michael Lindsey, *Direct interpolative construction of the discrete Fourier transform as a matrix product operator*, arXiv:2404.03182 (2024)

Step 1: Recursive decomposition of DFT

DFT = products of smaller DFTs with some phase modifications
Closely related to the Cooley–Tukey algorithm (fast Fourier transform)



$$
\begin{aligned}
&\mathbf{F}_n(s,t)\\
&= e^{-\pi i \sum_{k,l=1}^{n} 2^{-k} 2^{l} \sigma_k \tau_l}\\
&= e^{-\pi i \sum_{k,l=1}^{m} 2^{-k} 2^{l} \sigma_k \tau_l} \times e^{-\pi i \sum_{k=m+1}^{n} \sum_{l=1}^{m} 2^{-k} 2^{l} \sigma_k \tau_l}\\
&\quad \times e^{-\pi i \sum_{k,l=m+1}^{n} 2^{-k} 2^{l} \sigma_k \tau_l}\\
&= \mathbf{F}_m(\sigma_{1:m}, \tau_{1:m}) \mathbf{F}_{n-m}(\sigma_{m+1:n}, \tau_{m+1:n}) e^{-2\pi i x_{>m} y_{\leq m}}
\end{aligned}
$$

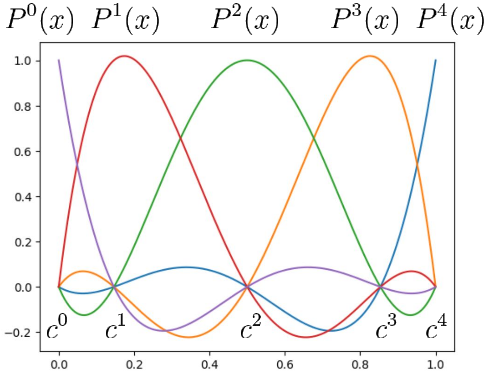
This part has low-rank decomposition, implying efficient QTT representation!

where we introduced new variables:

$$
x_{>m} = \sum_{k=m+1}^{n} 2^{m-k} \sigma_k \in [0,1], \quad y_{\leq m} = \sum_{l=1}^{m} 2^{l-m-1} \tau_l \in [0,1]
$$

Step 2: Low-rank decomposition by Lagrange interpolation

Lagrange interpolation on shifted Chebyshev-Lobatto grids:



$$f(x) \approx \sum_{\alpha=0}^{K} f(c^\alpha) P^\alpha(x)$$

$$c^\alpha = \frac{1}{2}\left[\cos\left(\frac{\pi(K-\alpha)}{K}\right) + 1\right]$$

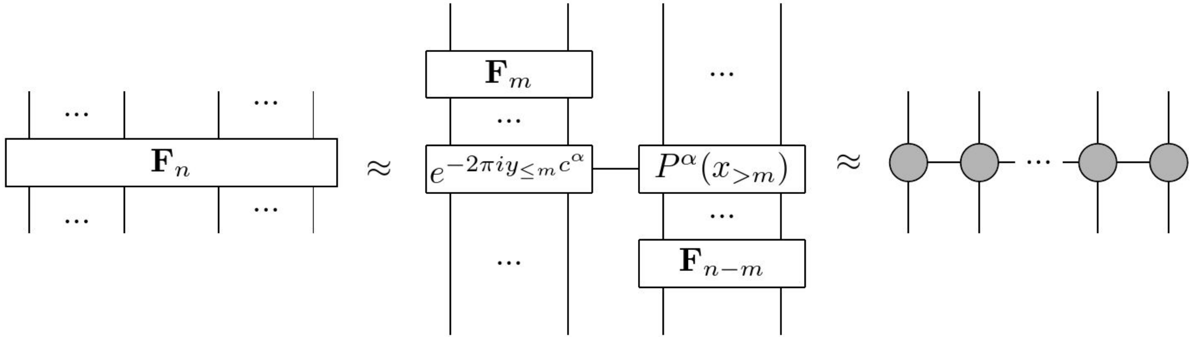
Entry-wise error from interpolation for our case:

Independent of n, m
i.e. true for all sizes & all bonds

$$\|e^{-2\pi i x_{>m} y_{\leq m}} - \sum_{\alpha=0}^{K} e^{-2\pi i y_{\leq m} c^\alpha} P^\alpha(x_{>m})\|_\infty \leq \frac{4\left(\frac{\pi}{2}\right)^{K+1} e^K K^{-K}}{K - \frac{\pi}{2}} =: E_K$$

Step 3: QTT with bounded Frobenius norm error

Bounded Frobenius norm error for every bond → there exists QTT with controlled Frobenius norm error [1]



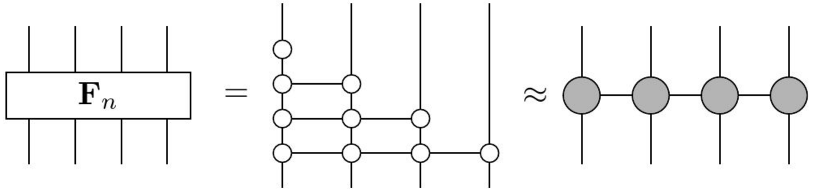
$$\forall m \; \frac{\|\mathbf{F}_n - \mathbf{F}'_{n,m}\|_F}{\|\mathbf{F}_n\|_F} \leq E_K \qquad \frac{\|\mathbf{F}_n - \mathbf{F}''_n\|_F}{\|\mathbf{F}_n\|_F} \leq \sqrt{n-1} E_K$$

Qualitatively same results in [2], except singular values are directly bounded

[1] I. V. Oseledets, *Tensor-Train Decomposition*, SIAM Journal on Scientific Computing 2011 33:5, 2295-2317
[2] JC, E.M. Stoudenmire, and Steven R. White, *Quantum Fourier Transform Has Small Entanglement*, PRX Quantum (2023) 4, 040318

How to construct such QTT?

1. TT-SVD (i.e. apply SVD sequentially on every bond) [1]
   - Gives QTT with Frobenius norm error $\sqrt{n-1}E_K$ (quasi-optimal)
   - Exponential time to construct
2. Expand DFT as *n x n* 2D tensor network and approximately contract [2]
   - Hard to analyze error; accurate in practice
   - $O(n^2K^3)$ time to construct



3. Recursive interpolation [3]
   - Controlled entry-wise error, but worse than optimal
   - Entries have analytical expressions, in principle no need to construct

[1] I. V. Oseledets, *Tensor-Train Decomposition*, SIAM Journal on Scientific Computing 2011 33:5, 2295-2317
[2] JC, E.M. Stoudenmire, and Steven R. White, *Quantum Fourier Transform Has Small Entanglement*, PRX Quantum (2023) 4, 040318
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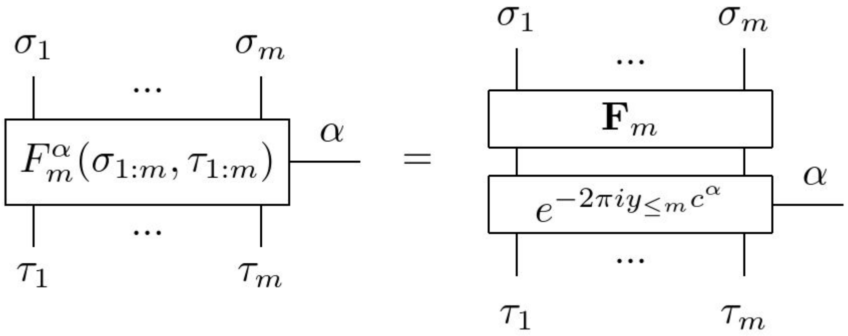
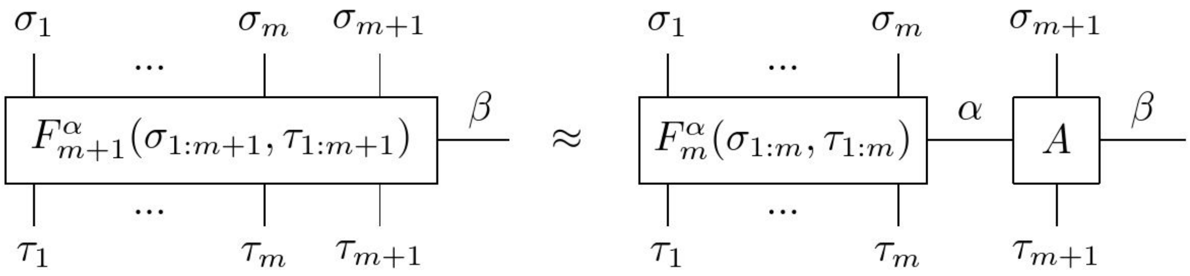
Analytical construction by recursive interpolation

Define the m-th interpolative tensor:



$$= e^{-\pi i \sum_{k,l=1}^{m} 2^{-k} 2^{l} \sigma_k \tau_l} e^{-2\pi i y_{\le m} c^\alpha}$$

Interpolate the (m+1)-th interpolative tensor with the m-th one:

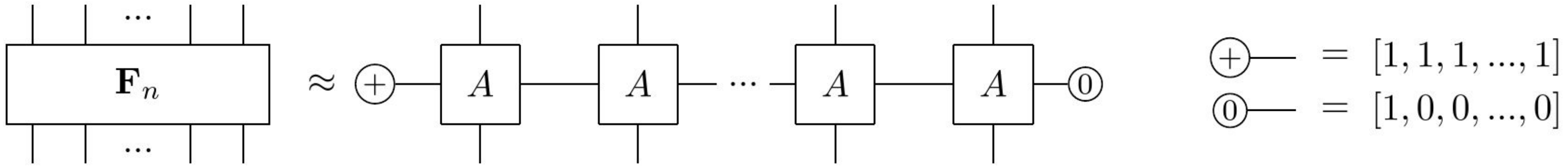


Solution:

$$A^{\alpha\beta}(\sigma, \tau)$$
$$= P^\alpha \left(\frac{\sigma + c^\beta}{2} \right) e^{-\pi i (\sigma + c^\beta)\tau}$$

Analytical construction by recursive interpolation

Gives QTT with translation-invariant sites and closed boundaries



Entry-wise error bound:

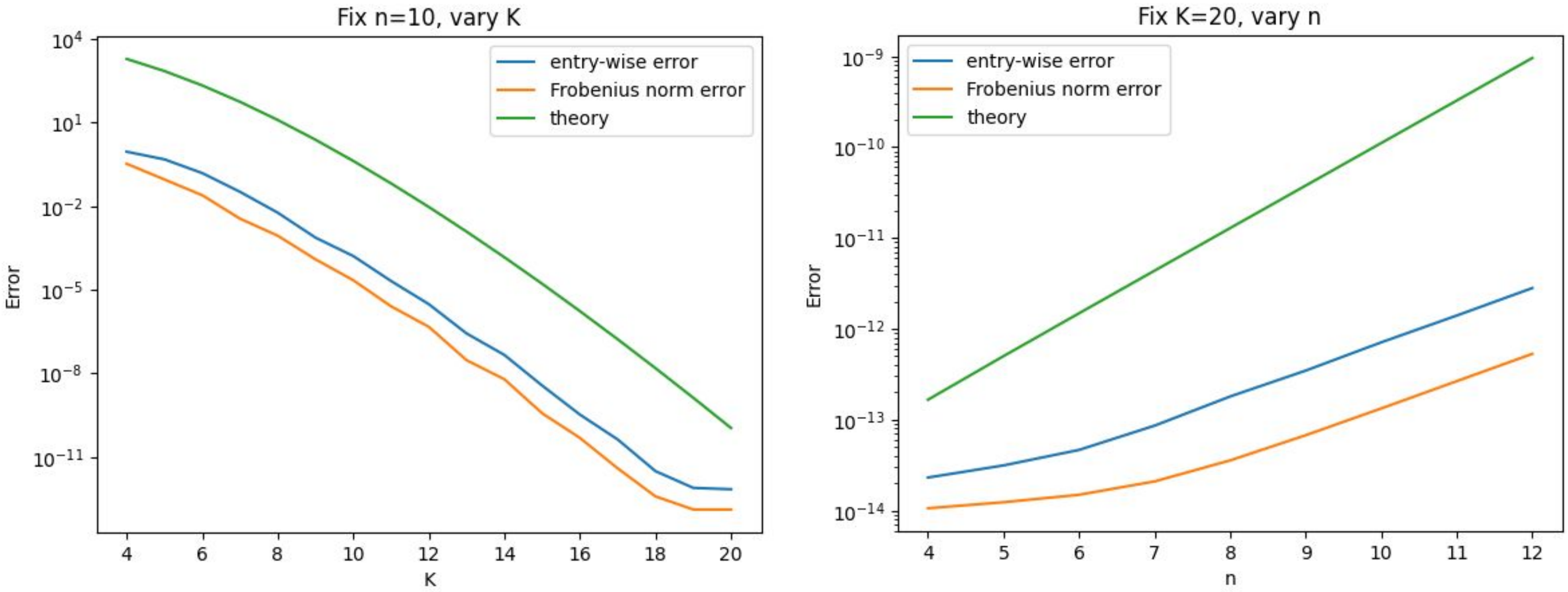
$$\max_{s,t} |\mathbf{F}_n(s,t) - \mathbf{F}''_n(s,t)| \leq \frac{\Lambda_K^{n-1} - 1}{\Lambda_K - 1} E_K$$

Worse than optimal

$$\Lambda_K \leq 1 + \frac{2}{\pi} \log(K+1)$$

Lebesgue constant for
Chebyshev-Lobatto grids

Numerics


Fix n=10, vary K
entry-wise error
Frobenius norm error
theory
Error
K
10^{4}
10^{1}
10^{-2}
10^{-5}
10^{-8}
10^{-11}
4
6
8
10
12
14
16
18
20
Fix K=20, vary n
entry-wise error
Frobenius norm error
theory
Error
n
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-14}
4
5
6
7
8
9
10
11
12

Conclusion

- We proved that DFT can be represented as a low-rank QTT
- We gave an analytical construction through interpolative decomposition

Discussion

- Tighter bounds?
- Generalize to other operators?
- Applications of the analytical construction?

Acknowledgement

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Thanks for discussion: Sandeep Sharma, Garnet Chan